Spacecraft Approaching Technique

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Abstract

Using already established equations of motion for particles under a gravitational force, we analyze the motion of two spacecrafts in the same circular orbit approaching each other if one of the craft were to speed up or slow down. The purpose of this paper is to analyze and develop equations of motion for the transfer of spacecrafts from circular orbits to elliptical orbits with the intention of spacecraft approach. We can come up with a table of velocities that allow for a number of safe approaches for the spacecraft to take. We found that projecting a safe approach requires that we know the new speed of the transfer spacecraft, the desired change in distance between the two spacecrafts, and the magnitude of the radius vector of the perigee for the elliptical transfer.

Introduction

Newton's second law of motion coupled with the principles of kinematics allows for the use of the methods of conservation of momentum and energy to relate force, mass, velocity and time. The application of conservation of momentum and energy in explaining the motion of a particle is extremely useful when dealing with conservative forces such as gravity, since acceleration no longer becomes a necessary relation to analyze the motion of said particle.

Walter Hohmann, in his 1925 book *The Accessibility of Celestial Bodies*, calculated an elliptical orbit transfer between circular coplanar orbits of different radii around a planet. Hohmann used half of an ellipse to accomplished the transfer. Hohmann's research on transfers between orbits around celestial bodies was a result of his fascination with outer space travel. He knew that to travel between and around planets, spacecrafts would require large amounts of fuel, so he discovered the most efficient way to transfer spacecrafts from a circular orbit to an elliptical orbit and then back to another circular orbit.

When in a circular orbit around a celestial body, if a spacecraft slows down or speeds up due to the thrust of its rockets, it enters an elliptical orbit with the point of thrust being the new apogee or perigee respectively. This paper analyzes a similar technique to what Hohmann theorized, except that the spacecraft entering the elliptical orbit will complete one or a number of orbits before re-entering its original circular orbit. The new period of rotation of the space craft in the elliptical orbit will be shorter than the period for the spacecraft still in the circular orbit, so that when the spacecraft returns to its original circular orbit, it will be closer to the other spacecraft.

Theoretical Application

The initial conditions of spacecrafts A and B are that both are moving in the same circular orbit. Spacecraft B will either slow down or speed up, thus entering an elliptical orbit. As shown in figure one Θ_1 is the angle to A when B reaches the fixed line L-L^1. Equations of motion state that the velocity required to have a circular orbit around a planet (Earth) is

$$v_{cir}=\sqrt{\frac{GM}{r_a}} \quad ,$$

and the period (P_A) of travel for the orbit is

$$P_A=\frac{2\pi r_a}{v_{cir}} \quad .$$

*A table of all notations can be found in Appendix 1

The final condition of spacecraft B is that it has ended the first elliptical orbit. Spacecraft A is closer to B. We first must express spacecraft B's period with respect to the spacecraft A's period as shown:

$$P_B=P_A-\frac{\theta_1-\theta_2}{\omega_A} \quad , \quad \Delta t=\frac{\Delta\theta}{\omega_A} \quad ,$$

$$P_B=P_A-\Delta t \quad .$$

The equations we are interested in that relate to a elliptical orbit around a planet are as follows:

$$P_B=\frac{2\pi ab}{h} \quad , \quad a=\frac{r_a+r_p}{2} \quad , \quad b=\sqrt{r_a r_p}$$

$$h=r_a v_a \quad .$$

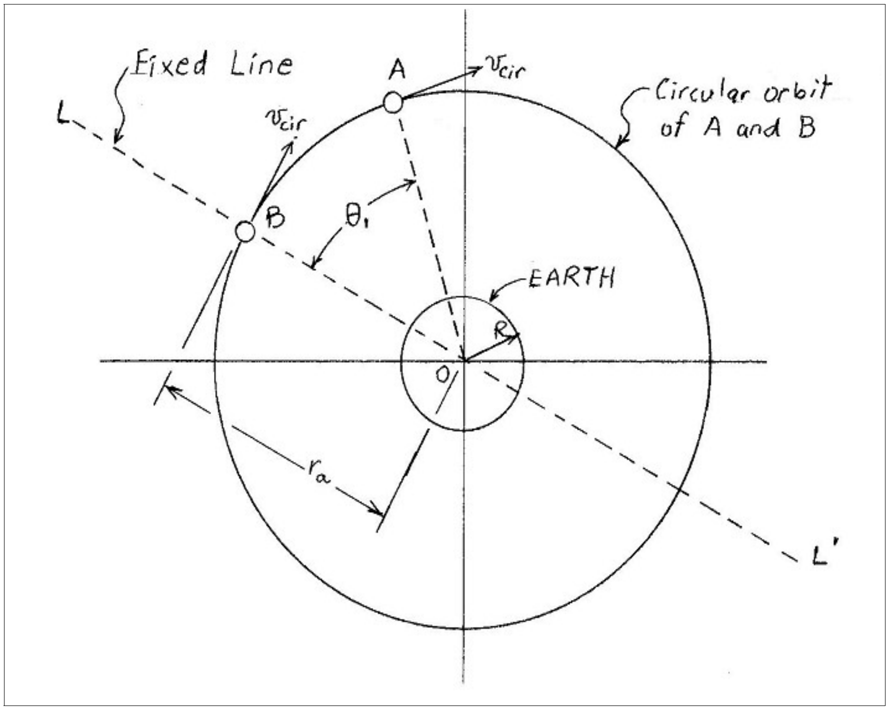

Figure 1. Spacecraft B is beginning its transfer to a full elliptical orbit around planet Earth.

To solve for the new velocity at the apogee, we apply the conservation of energy principle at the apogee and perigee,

$$E_a = E_p \quad ,$$

the points with the greatest and smallest distances from the focus (planet). As seen in Figure 2, sub-index *a* refers to the apogee and sub-index *p* refers to the perigee. We then expand the relation to distinguish kinetic from potential energy:

$$T_a + U_a = T_p + U_p$$

$$\frac{1}{2}v_a^2 - \frac{GM}{r_a} = \frac{1}{2}v_p^2 - \frac{GM}{r_p} \quad .$$

We then apply the principle of conservation of angular momentum at the apogee and perigee:

$$L_{o,a} = L_{o,p}$$

$$r_a m v_a \sin(\theta_a) = r_p m v_p \sin(\theta_p) \quad ,$$

where θ_a is equal to θ_b at 90 degrees.

The above equation represents the magnitude of the cross product between the radial vector and the velocity vector. For both the perigee and apogee the angle between these two vectors is ninety degrees so the magnitude of the total angular momentum at these two points is only dependent on the magnitude of the velocity and radius as shown :

$$r_a v_a = r_p v_p$$

Taking our equations of energy and angular momentum we substitute

$$v_p = \frac{r_a v_a}{r_p} \quad \text{into}$$

$$v_a^2 - v_p^2 = \frac{2\mathrm{GM}}{r_a} - \frac{2\mathrm{GM}}{r_p}$$

and solving velocity to obtain:

$$v_a = \sqrt{\frac{2\mathrm{GM}}{(r_a + r_p)} * \frac{r_p}{r_a}} \quad .$$

We now substitute circular, orbital velocity

$$v_{cir} = \sqrt{\frac{GM}{ra}} \quad \text{into} \quad P_A = \frac{2\pi r_a}{v_{cir}} \quad \text{to get}$$

$$P_A = \frac{2\pi r_a^{\frac{3}{2}}}{\sqrt{GM}} \quad .$$

Then by substituting P_A of the above equation into $P_B = P_A - \Delta t$ we obtain:

$$\frac{2\pi ab}{h} = \frac{2\pi r_a^{\frac{3}{2}}}{\sqrt{GM}} - \Delta t$$

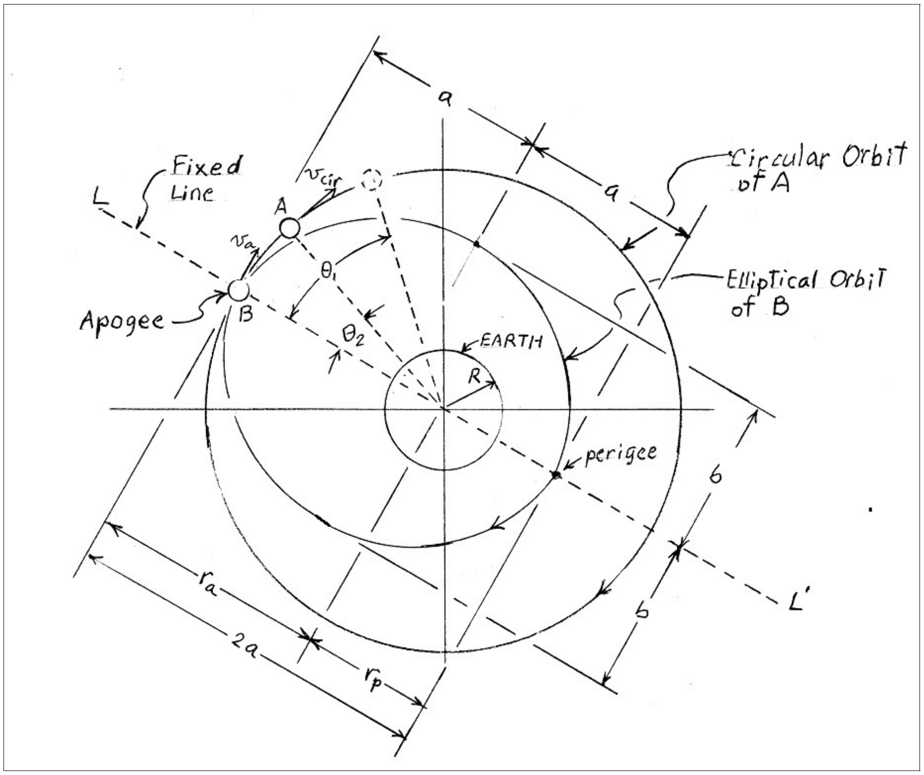

Figure 2. The final conditions of spacecraft B is that it has arrived to the origin of the full elliptical orbit. Spacecraft A is closer to spacecraft B now.

Now that we have only terms of Δt, r_p, and r_a

$$r_p = \left[\frac{\sqrt{2GM}}{\pi} \left(\frac{2\pi r_a^{\frac{3}{2}}}{\sqrt{GM}} - \Delta t \right) \right]^{\frac{2}{3}} - r_a \quad ,$$

we can solve for r_p. This simplifies the calculation for how close spacecraft B can approach a planet on its elliptical path so as to lessen the time lag between its circular orbit and the circular orbit of spacecraft A.

Analysis

We have made a graph of a family of three approaching spacecraft curves (Figure 3) for planet Earth at altitudes of 100, 200 and 300 km using different Δt of 4, 10, 20, 40, 60, 80, 100, 120, and 140s. We limit the extent of deceleration for spacecraft B to where it's perigee is just above the atmosphere of earth

$$R + atm_{depth} = 6.44\text{x}10^6\ m \quad .$$

There is a difference in time between the transition to an elliptical orbit and the transition to a circular orbit for a spacecraft. This is due to the nature of a rocket's propulsion, for mass is expended to increase or decrease in speed. We will show that this difference in mass has a minimal effect on the Approaching Technique depicted thus far.

Disclaimer-For simplicity in this paper the following assumptions hold:

1. The Change of speed at the apogee (because of the rocket firing) occurs in a very short period of time.
2. The radius at the apogee (r_a) remains constant during the rocket firing.
3. Only the rocket thrust is the unbalanced force during the rocket firing (virtually no gravity).

First, we analyze the decreasing in speed of spacecraft B at the apogee (Figure 4):
By applying newton's 2nd Law we can relate the reverse thrust of the
rocket to the force: $-Thrust = ma$.
We expand the relation into a differential and introduce the rate of burn into the equation

$$-\frac{dm}{dt} v_{rel} = m \frac{dv}{dt} , \frac{dm}{dt} = R_{burn} \quad .$$

Altitude	r_a	v_{cir}	Δt	r_p	v_a	Limitations
(km)	(m)	(m/s)	(s)	(m)	(m/s)	
100	6.47x10^6	7853	4	6.463x10^6	7851	
			10	6.453x10^6	7848	
			20	6.437x10^6	7843	
			40	6.403x10^6	N/A	$r_p < r_{p,min}$
200	6.57x10^6	7793	4	6.563x10^6	7791	
			10	6.553x10^6	7789	
			20	6.537x10^6	7784	
			40	6.504x10^6	7774	
			60	6.471x10^6	7764	
			80	6.437x10^6	7754	
			100	6.404x10^6	N/A	$r_p < r_{p,min}$
300	6.67x10^6	7735	4	6.663x10^6	7733	
			10	6.654x10^6	7730	
			20	6.637x10^6	7725	
			40	6.604x10^6	7716	
			60	6.571x10^6	7706	
			80	6.538x10^6	7696	
			100	6.505x10^6	7686	
			120	6.472x10^6	7676	
			140	6.439x10^6	N/A	$r_p < r_{p,min}$

Table 1. Table of Safe Approaches for Spacecraft B

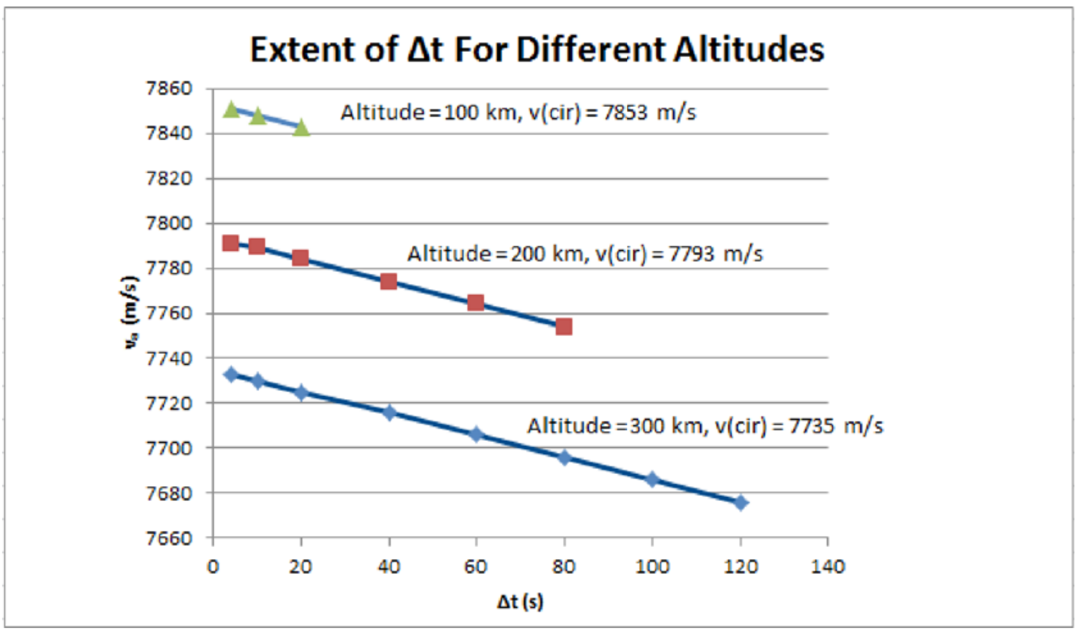

Figure 3. The graph shows that the maximum differences in periods are accomplished at higher altitudes. The maximum difference in periods at 300 km of altitude is six times greater than that at 100 km.

If we substitute the following expression into our equation

$$-R_{burn} v_{rel} = m \frac{dv}{dt} \quad ,$$

and then express the new mass in terms of the initial mass minus the mass burned in transition,

$$\left(-\frac{R_{burn}}{m}\right) v_{rel} = \frac{dv}{dt} \, , m = m_i - R_{burn} t \quad ,$$

then we arrive to a differential equation that can be analytically integrated:

$$\left(-\frac{R_{burn}}{m_i - R_{burn} t}\right) v_{rel} = \frac{dv}{dt} \quad .$$

We integrate the relation:

$$-R_{burn} v_{rel} \int \frac{dt}{m_i - R_{burn} t} = \int dv$$

$$v_{rel} \ln(m_i - R_{burn} t) = v + c \quad .$$

When t = 0

$$v = v_{cir} \, ; c = v_{rel} \ln(m_i) - v_{cir} \quad ,$$

and we can then obtain the initial conditions

$$v = v_{cir} + v_{rel} \ln\left(\frac{m_i - R_{burn} t}{m_i}\right)$$

For the final conditions, where spacecraft B is slowing down, we first find the time of thrust by making these substitutions $v = v_a \, , t = t_{th}$ and solving for t

$$v_a = v_{cir} + v_{rel} \ln\left(\frac{m_i - R_{burn} t_{th}}{m_i}\right)$$

$$t_{th} = \frac{m_i}{R_{burn}} \left(1 - e^{\frac{(v_a - v_{cir})}{v_{rel}}}\right) \quad .$$

The mass of the propellant used and the mass of spacecraft B at the end of the thrust are respectively:

$$m_{pro} = R_{burn} t_{th}$$

$$m_f = m_i - m_{pro} \quad .$$

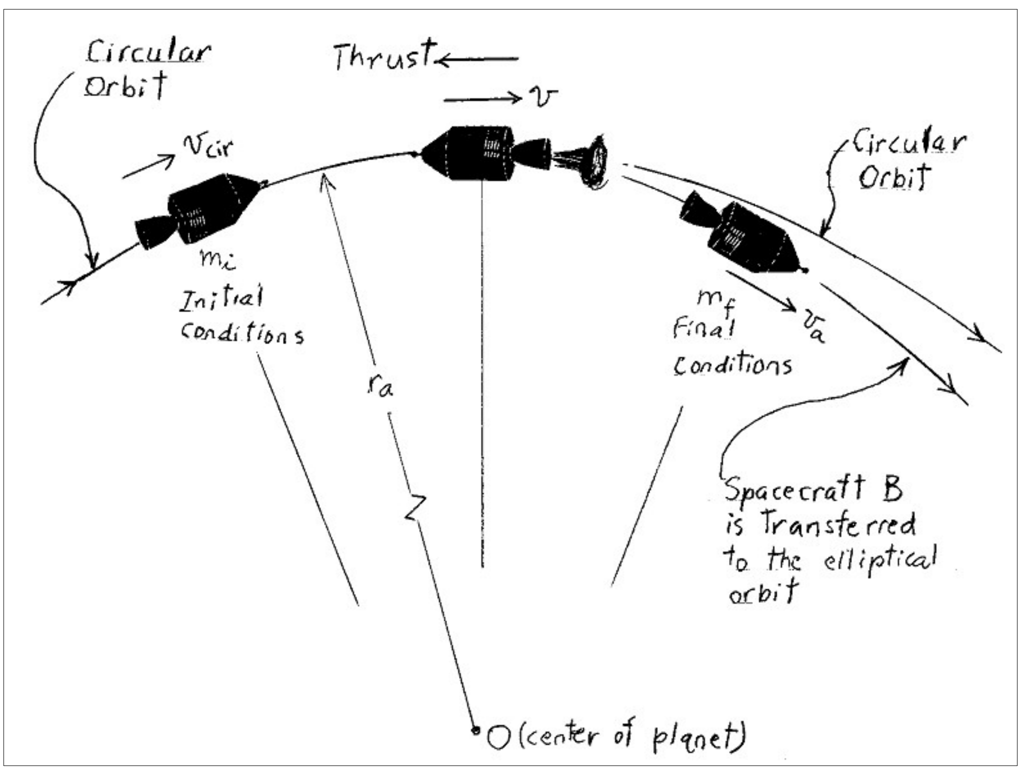

Figure 4. Spacecraft B is reducing its speed by firing the rocket in reverse position. The firing occurs between the initial and final conditions shown.

We use the negative expression for thrust when increasing the speed of spacecraft B at the apogee (Figure 5):

$$Thrust = ma$$
$$-v_{rel}\ln(m_i - R_{burn}t) = v + c \quad ,$$

where initial conditions are t=0, and $v=v_a$. At final conditions $v=v_{cir}$, $t=t_{th}$ and the circular velocity is:

$$v_{cir} = v_a + v_{rel}\ln\left(\frac{m_i}{m_i - R_{burn}t_{th}}\right) \quad .$$

We now solve for the time of thrust,as before, and obtain the similar equation

$$t_{th} = \frac{m_i}{R_{burn}}\left(1 - e^{\frac{(v_a - v_{cir})}{v_{rel}}}\right) \quad .$$

Calculation

The expressions analyzing time of thrust for an increase and a decrease of speed for spacecraft B are equal to each other. However, because of a difference in mass between the two transfers of the rocket, the time for each will not be the same.

We now look at a specific case where spacecrafts A and B are in a circular orbit 300 km above the Earth and spacecraft B is 400 km away from spacecraft A. Spacecraft B first slows down and enters an elliptical orbit. Spacecraft B then enters back into the original circular orbit at the apogee after completing two elliptical turns. At the end of this, spacecraft B will be 5 km away from spacecraft A. Spacecraft B has a mass(including propellant) of 1200 kg, a rate-of-burn of 4 kg/s, and a relative velocity of1500 m/s. Data for spacecraft A is not required for these set of calculations. The depth of the atmosphere is70km. The mass of the Earth(M) will be considered 5.98×10^{24} kg, the radius(R) 6.37×10^{6} m, and the gravitational constant(G) 6.673×10^{-11} Nm^2/kg^2.

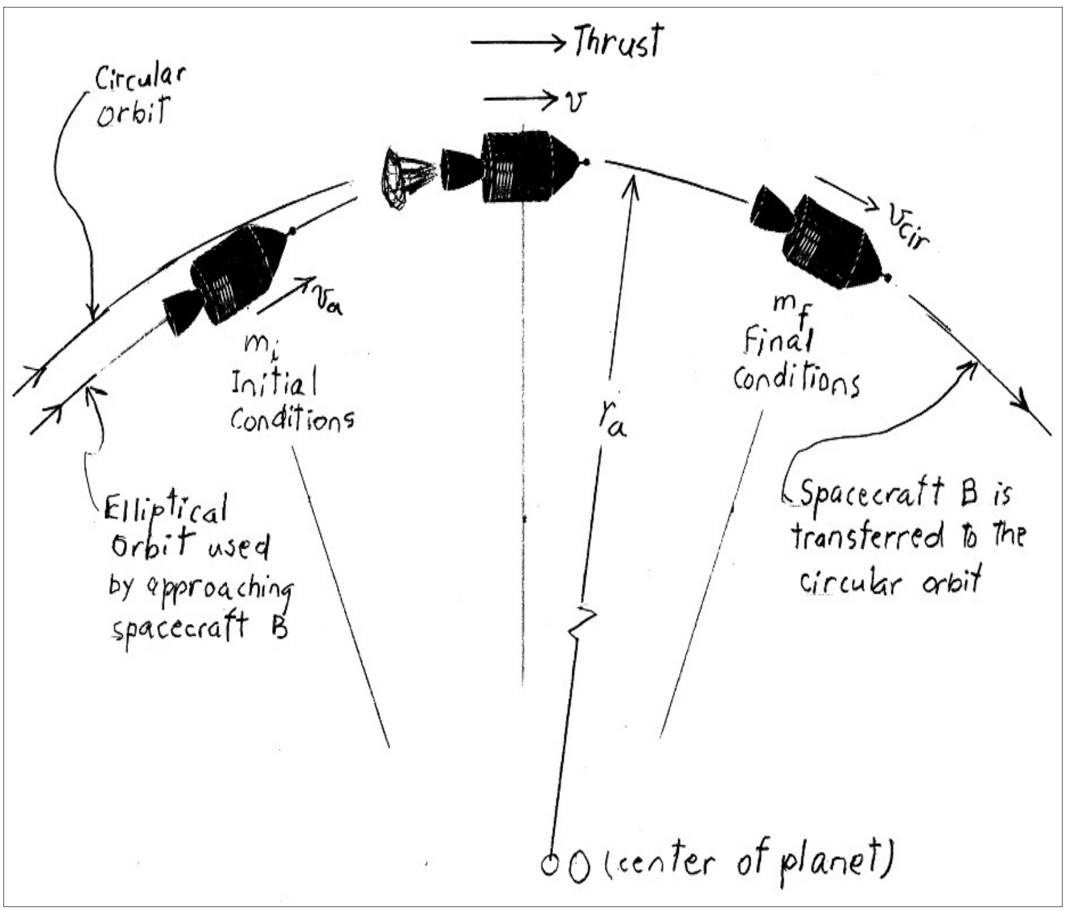

Figure 5. Spacecraft B is increasing its speed to enter back into the circular orbit by firing the rocket in the straight position. Firing occurs between initial and final conditions shown.

The magnitude of the radius vector of the apogee is

$$r_a = R + altitude = 6.67\text{x}10^6\,m \quad .$$

The circular velocity at 300 km altitude is:

$$v_{cir} = \sqrt{\frac{GM}{r_a}} = 7735\,m/s$$

The difference between the periods of the circular orbit and the elliptical orbit is:

$$\Delta t = \frac{d - d^1}{n_t v_{cir}}$$

Where d is the distance between A and B (400km), d^l is the final distance between A and B (5km), and n_t is the number of elliptical turns spacecraft B takes, and Δt is 25.533s.

The perigee radius vector is calculated to be

$$r_p = \left[\frac{\sqrt{2\text{GM}}}{\pi}\left(2\pi \frac{(6.67\text{x}10^6\,m)^{\frac{3}{2}}}{\sqrt{GM}} - 25.533\text{s}\right)\right]^{\frac{2}{3}} - 6.67\text{x}10^6\,m$$

$$r_p = 6.628058\text{x}10^6\,m \quad .$$

We then verify that spacecraft B does not encroach into the atmosphere when it is at the perigee:

$$r_{p,min} = R + atm_{depth} = 6.37\text{x}10^6\,m + 7\text{x}10^4\,m$$

$$r_{p,min} = 6.44\text{x}10^6\,m$$

$$r_p > r_{(p,\,min)}$$

$6.628058\text{x}10^6\,m > 6.440\text{x}10^6\,m$, the spacecraft does not reach the border of the atmosphere.

The speed of spacecraft B at the apogee is

$$v_a = \sqrt{\frac{2\text{GM}}{(6.670\text{x}10^6\,m + 6.628058\text{x}10^6\,m)} * \frac{6.62858\text{x}10^6\,m}{6.67\text{x}10^6\,m}}$$

$$v_a = 7722.3\,m/s \quad .$$

The time of thrust for spacecraft B to enter the elliptical orbit is

$$t_{th} = \frac{m_i}{R_{burn}}\left(1 - \text{e}^{\frac{(v_a - v_{cir})}{v_{rel}}}\right)$$

$$t_{th} = \frac{1200\,kg}{4\,kg/s}\left(1 - \text{e}^{\frac{(7722.3\,m/s - 7735\,m/s)}{1500\,m/s}}\right)$$

$$t_{th} = 2.53\,s \quad .$$

The time of thrust during transfer from circular to elliptical is greater than for elliptical to circular because of the loss of mass during the first transfer. This is shown in the next set of calculations. The new mass of spacecraft B after entering elliptical orbit is

$$m_f = m_i - m_{pro}$$

$$m_{pro} = R_{burn} t_{th} \quad , \quad m_{pro} = 10.12\,kg$$

$$m_f = 1189.88\,kg \quad .$$

The time of thrust for spacecraft B to enter back into original circular orbit is

$$t_{th} = \frac{1189.88\text{kg}}{4\text{kg}/s}\left(1 - \text{e}^{\frac{(7722.3\text{m}/s - 7735\text{m}/s)}{1500\,m/s}}\right) \quad ,$$

which as shown, is smaller than the thrust for the first transfer

$$t_{th} = 2.51\,s \quad < \quad t_{th} = 2.53\,s \quad .$$

The final mass of spacecraft B after entering original circular orbit is

$$m_{pro} = R_{burn} t_{th} \quad , \quad m_{pro} = 10.04\,kg$$

$$m_f = m_i - m_{pro} = 1189.88\text{kg} - 10.04\text{kg}$$

$$m_f = 1179.84\,kg \quad .$$

Conclusion

The effects of fuel lost from the two transfers creates a total time difference of 0.02 seconds between the predicted period for the elliptical orbit of B and the circular orbit of A . This time difference depends on the rate of burn, R_{burn}, and is small for our set of data. As to the effectiveness of the approaching technique, this can be summarized by a relation we will call the **Sensitivity of Approach:**

$$Sensitivity\ of\ Approach = \frac{\frac{d-d^1}{n_t}}{v_{cir}-v_a} = \frac{\frac{400\text{km}-5\text{km}}{2}}{7735\text{m}/s-7722.3\text{m}/s} = 15.55\,\frac{km}{ms^{-1}}$$

The term refers to the difference in distance between spacecraft A and B before and after the technique per the speed reduction of spacecraft B. The distance between spacecrafts A and B is reduced in 15.55km per 1 m/s in speed reduction of spacecraft B. This sensitivity of approach is very high for our sample calculations, and this method of approach is recommended for spacecrafts separated at great distances and only as a previous step to final coupling.

Appendix I

R, M = radius and mass of the Earth in *m, kg*
G = gravitational constant, 6,673x10^{-11} *Nm²/kg²*
v_{cir} = speed of spacecraft in a circular orbit in *m/s*
r_a = longer radius vector in *m*
r_p = shorter radius vector in *m*
P_A = period of spacecraft A in *s*
a, b = Semimajor and semiminor axis of the elliptical orbit in *m*
P_B = period of spacecraft B in *s*
ω_A = angular speed of spacecraft A in *rad/s*
Δt = difference between spacecraft periods in *s*
h = angular momentum per unit of mass of spacecraft B with respect to O, in m^2/s
v_a = speed (at the apogee) that should be given to spacecraft B to transfer to an elliptical orbit, in *m/s*
m = mass of spacecraft B in *kg*
m_i, m_f = initial and final masses of spacecraft B during any transfer, in *kg*
Thrust = push created by operating rocket of spacecraft B, in *N*
a = acceleration of spacecraft B in m/s^2
t = time in *s*
R_{burn} = rate of propellant burning in *kg/s*
v_{rel} =speed of exhaust propellant gases relative to spacecraft B, in *m/s*
t_{th} = duration of the thrust created by the exhausting gases, in *s*
m_{prop} = mass of the burned propellant during any transfer, in *kg*
d = initial distance between spacecraft A and B in *m*
d^1 = final distance between spacecraft A and B in *m*
n_t = number of elliptical turns of spacecraft B
atm_{depth} = depth of atmosphere in m

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