Particular Solution to a Time-Fractional Heat Equation

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Abstract

When the derivative of a function is non-integer order, e.g. the 1/2 derivative, one ventures into the subject of fractional calculus. The time-fractional heat equation is a generalization of the standard heat equation as it uses an arbitrary derivative order close to 1 for the time derivative. We present a particular solution to an initial-boundary-value time-fractional heat equation problem and compare the properties of the solutions when the time derivative order is varied. Orders of the time-fractional heat equation which return solutions that display physically impossible characteristics are also considered. One observed property is slightly less exponential decay of heat for the solutions of non-integer order time derivatives greater than one. Another property is solutions with non-integer order time derivatives less than one exhibit slightly greater exponential decay. Both of these comparisions are made with respect to the standard heat equation solution, where the order of the time derivative is one.

Keywords: Fractional Calculus; Time-Fractional Heat Equation; Time-Fractional Diffusion Equation.

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1 Introduction

In the last few decades, fractional differential equations have been found to be successful models of real life phenomenon (see e.g. [1, 2, 4] and their contained references). Motivated by the quote, "...we may say that Nature works with fractional time derivatives."[7], we investigate the initial-boundary-value problem (IBVP) for a time-fractional heat equation. Ultimately we seek to graph the solutions for when the standard heat equation, $D_t u = D_x^2 u$, is generalized to the time-fractional heat equation, $D_t^\alpha u = D_x^2 u$, for several values of α.

We proceed by first providing the overview for the IBVP, and solving it with the standard heat equation. Next, we solve the same IBVP with the generalized time-fractional heat equation, and solutions for multiple derivative orders are graphed for comparison with the standard solution. Finally, the solutions displaying physically unrealistic properties are plotted.

2 IBVP Heat Equation Solution

The standard heat equation is $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. For this equation the temperature is represented by u, which is a function of time, t, and space, x. To make comparison easier, we use the derivative operator D to write the heat equation with the following notation:

$$D_t u = D_x^2 u. \tag{1}$$

Consider an ideal one-dimensional rod of length L with the boundary conditions

$$u(t,0) = u(t,L) = 0 \tag{2}$$

for $t \geq 0$ and an initial condition with a parabolic profile of heat distribution given by

$$u(0,x) = -\frac{4a}{L^2}x^2 + \frac{4a}{L}x \tag{3}$$

shown in Figure 1. Note the initial condition satisfies the boundary conditions and has a maximum temperature of $u(0, L/2) = a$.

The solution is then assumed to be of the form $u(t,x) = T(t)X(x)$. Plugging this into

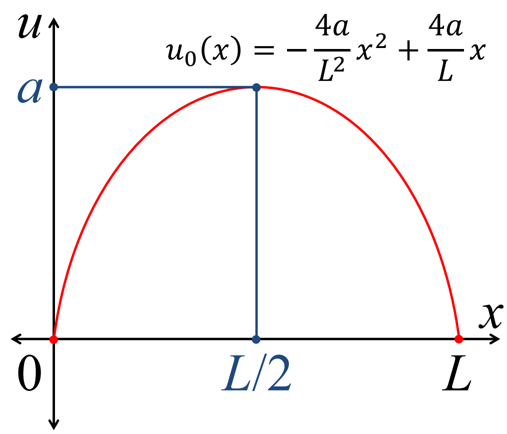

Figure 1: *Initial heat distribution along the rod.*

equation (1) yields $T'(t)X(x) = T(t)X''(x)$. Since this equation is separable, there exists a constant of separation, $C \in \mathbb{R}$, such that

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = C. \tag{4}$$

To obtain a non-trivial, real solution, assume $C < 0$. This will also satisfy the property of temperature decay to the defined boundary conditions. Thus we let $C = -\lambda^2$ for $\lambda \in \mathbb{R}$ and we must solve the ordinary differential equations (ODEs):

$$\begin{cases} X''(x) = -\lambda^2 X(x) \\ T'(t) = -\lambda^2 T(t). \end{cases} \tag{5}$$

These ODEs have the solutions $X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$ and $T(t) = C_3 e^{-\lambda^2 t}$. Reassigning the constants leads to the general solution to the heat equation (1)

$$u(t,x) = C_1 \cos(\lambda x) e^{-\lambda^2 t} + C_2 \sin(\lambda x) e^{-\lambda^2 t}. \tag{6}$$

Moving onto the particular solution, we start by satisfying the boundary conditions (2). By inspection, the condition $u(t,0) = 0$ requires $C_1 = 0$. The only non-trivial way to satisfy $u(t,L) = 0$ is by solving $\sin(\lambda L) = 0$. Thus

$$\lambda_n = \frac{n\pi}{L}, \tag{7}$$

for $n \in \mathbb{N}$, yields infinitely many solutions to the heat equation. The sum of such solutions is given by the Fourier series solution

$$u(t,x) = \sum_{n=0}^{\infty} C_n \sin(\lambda_n x) e^{-\lambda_n^2 t}. \tag{8}$$

The initial condition (3) is used to determine the Foureir coefficients (C_n) given by

$$C_n = \frac{2}{L} \int_0^L \left(-\frac{4a}{L^2} x^2 + \frac{4a}{L} x \right) \sin(\lambda_n x)\, dx. \tag{9}$$

Integrating and using the properties of sine and cosine will yield

$$C_n = \begin{cases} 0, & n \text{ even} \\ \dfrac{32a}{n^3 \pi^3}, & n \text{ odd}. \end{cases} \tag{10}$$

Ignoring the zero-valued even-indexed terms in equation (8), we have the particular solution to the IBVP heat equation (1-3):

$$u(t,x) = \sum_{n \in \mathbb{N},\ \text{odd}} \frac{32a}{n^3 \pi^3} \sin(\lambda_n x) e^{-\lambda_n^2 t}, \tag{11}$$

where we now have

$$\lambda_n = \frac{n\pi}{L}. \tag{12}$$

The main properties of this solution do not change for differing values of maximum initial heat a and rod length L. Using $a = L = 1$, the particular solution can be seen in Figure 2.

3 IBVP Time-Fractional Heat Equation Solution

The time-fractional heat equation generalizes the time derivative of the standard heat equation. Consider the equation

$$D_t^{\alpha} u = D_x^2 u \tag{13}$$

where $\alpha \in [1-\delta, 1+\delta] \subset \mathbb{R}$ and $\delta > 0$ is relatively small. From Luchko [5], uniqueness and existence are proven with similar general conditions to what we are considering. However,

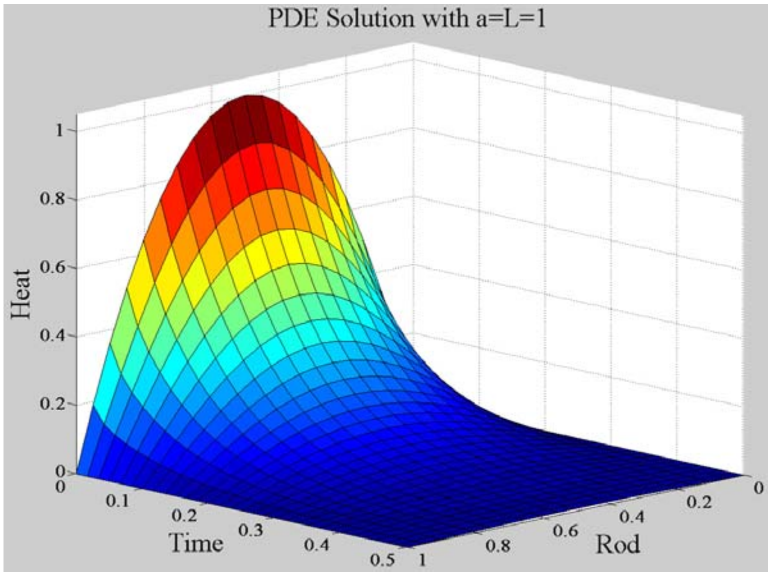

Figure 2: *A graph of the particular solution to the IBVP heat equation with* $a = L = 1$.

we emphasize that we seek to visualize what the differences in the solutions for equation (13) when α is varied.

Consider the same boundary and initial conditions (2-3) as well as a solution of the form $u(t, x) = T(t)X(x)$. Then the procedure of finding the solution does not change until we solve the differential equations (DEs)

$$\begin{cases} X''(x) = -\lambda^2 X(x) \\ D_t^\alpha T(t) = -\lambda^2 T(t), \end{cases} \tag{14}$$

which are analogous to equations (5). From Miller and Ross [3] or Luchko [5], the solution to the fractional differential equation (FDE) $D_t^\alpha T(t) = -\lambda^2 T(t)$ is the Mittag-Leffler function:

$$T(t) = \sum_{k=0}^{\infty} \frac{\left(-\lambda^2 t^\alpha\right)^k}{\Gamma(\alpha k + 1)}, \tag{15}$$

where Γ is the standard gamma function.

The general solution can be written at this point, but note from Section 2 all the unknowns of the IBVP are determined by the X portion of the equation. Therefore only the exponential part in solution (11) changes (to the Mittag-Leffler function), and we have the particular solution

$$u(t,x) = \sum_{n\in\mathbb{N},\ \text{odd}} \left[\frac{32a}{n^3\pi^3} \sin(\lambda_n x) \sum_{k=0}^{\infty} \frac{\left(-\lambda_n^2 t^\alpha\right)^k}{\Gamma(\alpha k+1)} \right], \tag{16}$$

where

$$\lambda_n = \frac{n\pi}{L}. \tag{17}$$

4 Solution Comparison

Now that we have the particular solution to the time-fractional heat equation (16), a plot comparing its solutions for various α values is desired. Unfortunately the Mittag-Leffler function part of the solution, equation (15), converges slowly. The convergence of this portion of our solution is slow enough that it computationally diverges for almost any $t > 0$. In lieu of using high precision numerical calculations, a "less rigorous" solution was found.

The pattern of the standard derivative of the exponential function $D_t^n\left[e^{rt}\right] = r^n e^{rt}$ where $n \in \mathbb{N}$ can be generalized to the fractional derivative as

$$D_t^\alpha\left[e^{rt}\right] = r^\alpha e^{rt}, \tag{18}$$

where $\alpha \in \mathbb{R}$, see [6] and contained references. Using this derivative and recalling the FDE is $D_t^\alpha T(t) = -\lambda_n^2 T(t)$, it can be seen that

$$T(t) = e^{\sqrt[\alpha]{-\lambda_n^2}\,t} \tag{19}$$

is a solution. By substituting solution (19) into equation (16), we have a new solution which can be numerically computed. Our alternate solution is

$$u(t,x) = \sum_{n\in\mathbb{N},\ \text{odd}} \frac{32a}{n^3\pi^3} \sin(\lambda_n x) e^{\sqrt[\alpha]{-\lambda_n^2}\,t}, \tag{20}$$

where λ_n is as in equation (17).

Note the potential problem of a negative number inside an even indexed root in the expression $e^{\sqrt[\alpha]{-\lambda_n^2}\,t}$. This leads to a potentially imaginary and non-unique solution. Since we are

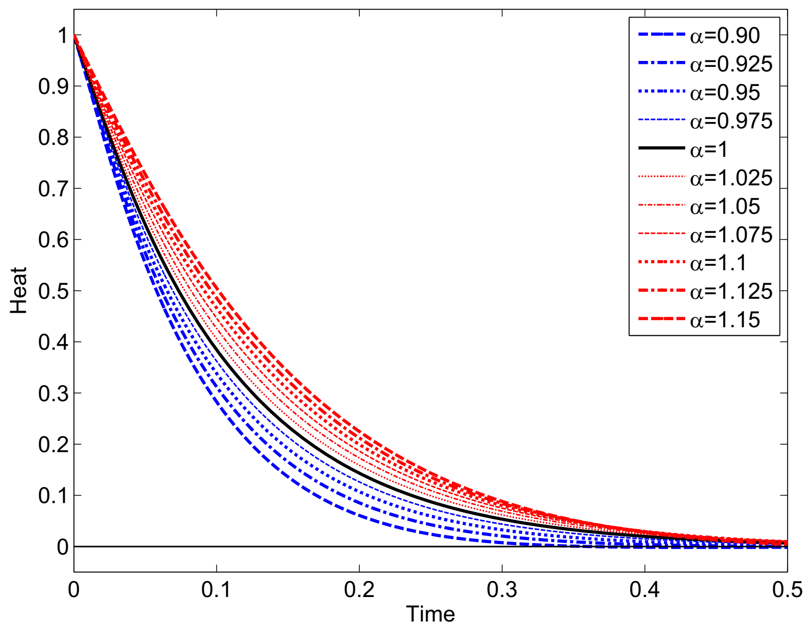

Figure 3: *Heat decay from the IBVP time-fractional heat equation at $x = L/2$ with $a = L = 1$.*

mainly interested in comparing the graphs of the solutions for multiple α values, we ignored the non-uniqueness property by only evaluating the real part of this expression. We were then able to create Figure 3, which is the heat decay for the rod at $x = L/2$ for several α values. All of the plotted solutions in Figure 3 are "physical realistic," in that they satisfy the boundary conditions, are free of negative temperature values, and don't display an increase in heat of the rod in the absence of any external heating factors.

In addition to the physical solutions plotted in Figure 3, there are several solutions outside of this range which provide physically unrealistic solutions. Figure 4 shows a plot of these various solutions.

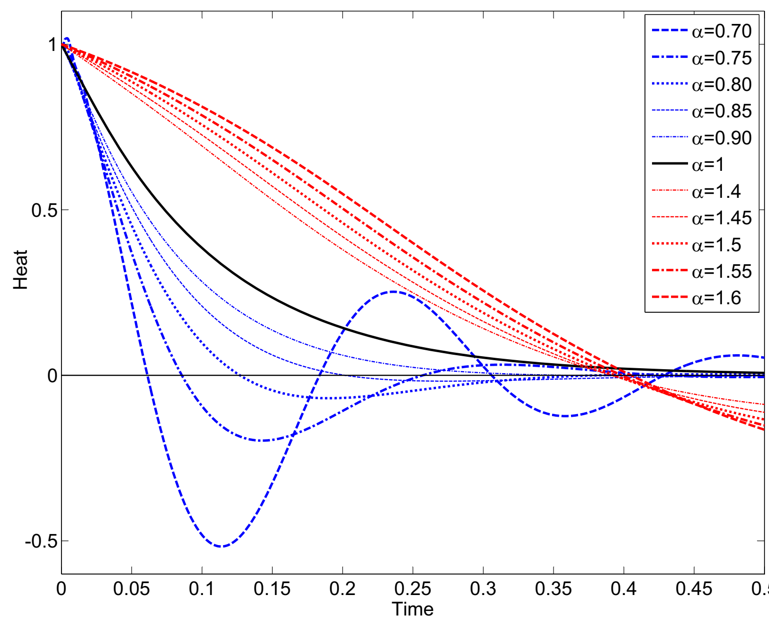

Figure 4: *Solutions for heat decay from the IBVP time-fractional heat equation at* $x = L/2$ *with* $a = L = 1$ *which display physically unrealistic properties.*

5 Conclusions and Future Work

From Figure 3, there are several important properties to be seen:

- A continuous change in α appears to yield a continous change in the heat decay profile of u.
- For the values $\alpha > 1$, there is slightly less exponential decay in heat compared to the heat decay for $\alpha = 1$.
- For the values $\alpha < 1$, there is slightly greater exponential decay in heat compared to the heat decay for $\alpha = 1$.
- All solutions sufficiently close to α satisfy the boundary conditions, and display physically realistic properties.

Therefore we conclude that the time-fractional heat equation is a physically legitimate generalization of the standard heat equation that might be used for values $\alpha \approx 1$. As expected however, when the value of α strays too far from $\alpha = 1$ the time-fractional heat equation will exhibit properties which are no longer realistic for describing the heat decay of an object.

Ideas for future work:

- Work with equation (16) to get the Mittag-Leffler portion of the solution to numerically converge. Then one can compare the unique solution (16) and confirm that the solution (20) is a viable alternative to work with.
- Do equations (16) or (20) solve the wave equation when $\alpha = 2$? If so, what are the IVBP conditions for the wave equation to have one of these as a solution? That is to say, can the heat equation "morph" into the wave equation through the time-fractional generalization?
- Solve the FDE $D_t^\alpha u = kD_x^\beta u$. The solution of this can be used to generalize the heat equation, the Laplace's equation, and the wave equation for appropriate values of α, β and k (near there classical values) and initial/boundary conditions.
- Does a fractional heat equation solution compare to empirical data with better accuracy/precision than the standard heat equation solution under certain conditions or with certain materials?

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