RELATIVISTIC PENDULA

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ABSTRACT

We solve for the angular motion of the relativistic simple pendulum driven by uniform driving forces of two distinct types: gravity and a uniform electric field. In both cases, we calculate and plot the angular motion of the pendulum in time. We also find the period of the pendulum as a function of its amplitude. While these two relativistic pendula do not differ greatly in their motion, the motion of each type of relativistic pendulum differs significantly from the classical large amplitude pendulum. The classical and relativistic pendula differ greatly in the period vs amplitude characteristics. We also calculated the period of each type of pendulum in the non-inertial frame of the pendulum itself. For a given amplitude, the period in the non-inertial frame is less than that determined in the stationary frame.

INTRODUCTION

One of the basic mechanics problems than an undergraduate physics or engineering major learns to solve is the motion of a simple pendulum. [1] Initially, the pendulum problem is solved assuming that the amplitude is small, resulting in simple harmonic motion. Later on, the student learns to solve Newton's law determining the motion of the pendulum for arbitrary amplitude, [2] a nonlinear differential equation whose solution is found in terms of elliptic functions.

The solution of the generalization of Newton's second law to account for the relativistic motion of a pendulum requires no essentially new techniques, but the resulting equations are a bit difficult to solve than those for the classical case. There are two such pendula whose motions are distinguished from one another by the nature of the uniform force which drives the pendulum: a constant gravitational field interacting with the mass at the end of the pendulum or a constant electric field interacting with a charged mass at the end of the pendulum. In both cases, one can find the period of the pendulum as a function of the proper amplitude by reducing the equations of motion to quadrature. This can be done both in a stationary inertial reference frame and the non-inertial reference frame of the pendulum mass itself.

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EQUATION OF MOTION OF THE RELATIVISTIC PENDULUM.

A plane simple pendulum of mass m, carrying charge q, is supported by a massless rod of proper length , L as shown in Figure 1. The pendulum is released from rest with respect to a fixed support (the origin of the inertial reference frame 'O') from angular position $\theta(0) = A$ as measured from the vertical. $\vec{W}(v)$ is a spatially uniform force in 'O' in the $-\hat{y}$ direction.

In reference frame 'O', Newton's second law is

$$\frac{d}{dt}\left(m\,\vec{v}\,\gamma\right) = \vec{T} + \vec{W}(v) \ , \tag{1}$$

where $\vec{T}$ is the tension in the rod and

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \ . \tag{2}$$

Taking the derivative in equation 1 and taking the cross product of each term in Equation 1 with $\vec{r}$, the position of the mass with respect to the origin of the 'O' reference frame yields:

$$m\,\dot{\gamma}\ \vec{r} \times \vec{v} + m\,\gamma\,\vec{r} \times \dot{\vec{v}} = \vec{r} \times \vec{W}(v) \ , \tag{3}$$

where the overdot denoted differentiation with respect to time. Since the tension in the rod is along the rod:

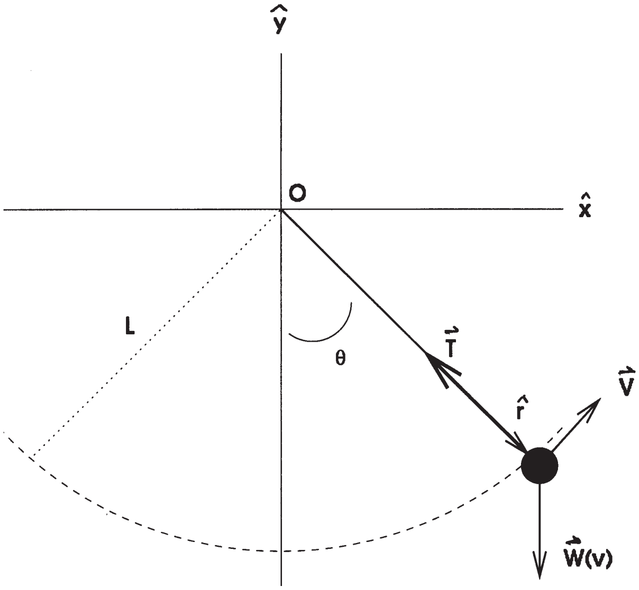

Figure 1
Schematic diagram of the pendulum showing the forces as determined in the inertial frame stationary with respect to the suspension point of the pendulum. The z direction points out of the diagram.

$$\vec{r} \times \vec{T} = 0 \ , \tag{4}$$

The remaining vector products are computed using the directions shown in Figure 1:

$$\begin{aligned} \vec{r} \times \vec{W}(v) &= -L\, W(v) \sin(\theta)\, \hat{z} \\ \vec{r} \times \left(m\, \dot{\gamma}\ \vec{v}\right) &= m\, \dot{\gamma}\ L\, v\, \hat{z} \\ \vec{r} \times \left(m\, \gamma\, \dot{\vec{v}}\right) &= m\, \gamma\, L\, \dot{v}\, \hat{z} \ , \end{aligned} \tag{5}$$

where $\hat{z}$ is the unit vector in the z-direction. The third equation in Equation 5 follows from the identity:

$$\vec{r} \times \dot{\vec{v}} = \frac{d}{dt}\left(\vec{r} \times \vec{v}\right) = \frac{d}{dt}\left(L\, v\, \hat{z}\right) \ . \tag{6}$$

Inserting Equation 5 into equation 3 yields:

$$v\, \dot{\gamma} + \gamma\, \dot{v} = -\frac{W(v)}{m} \sin(\theta) \ . \tag{7}$$

Differentiating Equation 2 with respect to time gives:

$$\dot{\gamma} = \frac{v\, \dot{v}}{c^2}\, \gamma^3 \ , \tag{8}$$

Putting Equation 8 into Equation 7 gives:

$$\gamma^3\, \dot{v} = -\frac{W(v)}{m} \sin(\theta) \ . \tag{9}$$

To bring Equation 9 into a more standard form, the dependent variable is changed from *v(t)* to θ*(t)*. Since *v* is the tangential velocity of the mass, it is related to the angular position by:

$$v = L\, \dot{\theta} \quad \text{and} \quad v = L\, \ddot{\theta} \tag{10}$$

Substituting Equation 10 into Equation 9 :

$$\ddot{\theta} = -\frac{W(\dot{\theta})}{m\, L} \sin(\theta) \left(1 - \frac{L^2\, \dot{\theta}^2}{c^2}\right)^{\frac{3}{2}} \tag{11}$$

We note in passing that Equation 11 reduces to the equation governing the non-relativistic pendulum as *v/c* -> 0 and the driving force replaced by *W = mg*.

It is convenient to introduce a dimensionless time variable:

$$\eta \equiv \frac{c\, t}{L} \tag{12}$$

Changing the variables according to Equation 12, Equation 11 becomes:

$$\frac{d^2\theta}{d\eta^2} = -\frac{L\, W\!\left(\frac{d\theta}{d\eta}\right)}{m\, c^2} \left(1 - \left(\frac{d\theta}{d\eta}\right)^2\right)^{\frac{3}{2}} \sin(\theta) \tag{13}$$

Equation 13 determines the motion of the pendulum in the reference frame ‘O’ due to an arbitrary driving force *W(v)*. While the equation is nonlinear, it is still an autonomous differential equation which can be reduced to quadrature by standard techniques. [4] We solve Equation 13 for the two special cases of the driving force and determine the angular motion and period in each case.

MOTION AND PERIOD

Motion

There are two distinct motions of the relativistic pendulum. The first type is driven by a uniform gravitational field. This field is effectively velocity dependent because the effective gravitational mass of the pendulum $E/c^2 = m\gamma$. Thus the driving force takes on the form:

$$\vec{W}(v) = m\, g\, \gamma\, (-\hat{y}) \tag{14}$$

The second type of relativistic pendulum is driven by a uniform, constant electric field, *E*. Because the total charge *q* is a Lorentz invariant scalar, this force does not depend on the speed of the pendulum. Ignoring any radiation due to the acceleration of the charged pendulum, the driving force becomes:

$$\vec{W}(v) = q\, E\, (-\hat{y}) \tag{15}$$

The equations of motion for each of these special cases follow from Equation 13 by inserting the appropriate form of the driving force (Equations 14 and 15). The gravity driven pendulum equation is denoted by *(G)* and the electric field driven pendulum equation by *(E)*:

$$\begin{aligned} \frac{d^2\theta}{d\eta^2} &= -\kappa \sin(\theta) \left(1 - \left(\frac{d\theta}{d\eta}\right)^2\right), \quad (G) \\ \frac{d^2\theta}{d\eta^2} &= -\kappa \sin(\theta) \left(1 - \left(\frac{d\theta}{d\eta}\right)^2\right)^{\frac{3}{2}}, \quad (E) \end{aligned} \tag{16}$$

where we have defined the constant κ by:

$$\begin{aligned} \kappa &= \frac{m\, g\, L}{m\, c^2} \quad (G) \\ \kappa &= \frac{q\, E\, L}{m\, c^2} \quad (E) \end{aligned} \tag{17}$$

To numerically solve Equation 16 for θ(η), we convert these equations of motion to two systems of first-order differential equations:

$$\frac{d\theta}{d\eta} = \Omega,$$
$$\frac{d\Omega}{d\eta} = F(\theta,\Omega) \ , \qquad (18)$$

where the 'forcing function" , F(θ, Ω) is:

$$F(\theta,\Omega) = -\kappa \sin(\theta)\left(1-\Omega^2\right) \ , \quad (G)$$
$$F(\theta,\Omega) = -\kappa \sin(\theta)\left(1-\Omega^2\right)^{\frac{3}{2}} \ , \quad (E) \qquad (19)$$

The coupled first order differential equations, Equations 18 determining θ(η) were solved using the second-order Runge-Kutta approximation in MAPLEV™. 5 The numerical solutions for generated using this method are plotted in Figure 2. We chose the amplitude, A = 3 radians and the constant κ = 1 in both cases. The value of A = 3 radians corresponds to a very large initial amplitude, so even in the non-relativistic case, the equation of motion is nonlinear. Three cases for these initial parameters are shown in Figure 2, both driving forces in the relativistic case and the non-relativistic gravitationally driven pendulum. There is a significant difference between the relativistic and non-relativistic motion, but with the parameter κ = 1, there is not a great difference between the gravity driven and electric field driven relativistic pendula over the entire period of the motion.

Period

The equations of motion, Equations 16, are autonomous differential equations, the independent variable is not

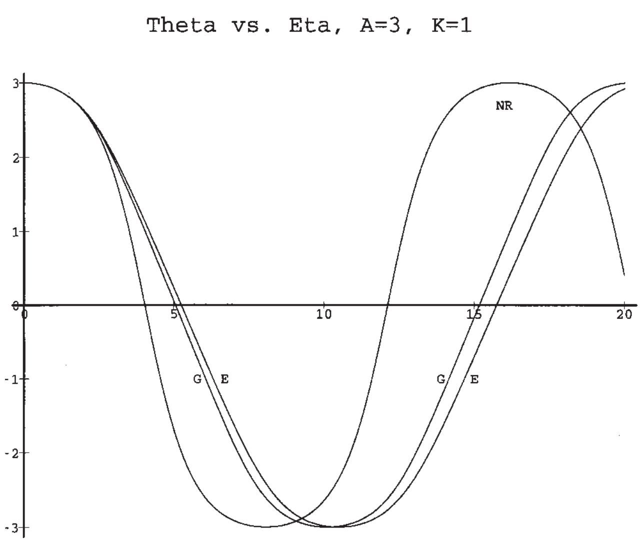

Figure 2
Angular position (in radians) of the pendulum vs time: non-relativistic, (NR); relativistic gravity driven, (G); relativistic electric field driven, (E). The horizontal axis is the dimensionless time $\eta = ct/L$. This time is that determined with respect to a stationary inertial frame.

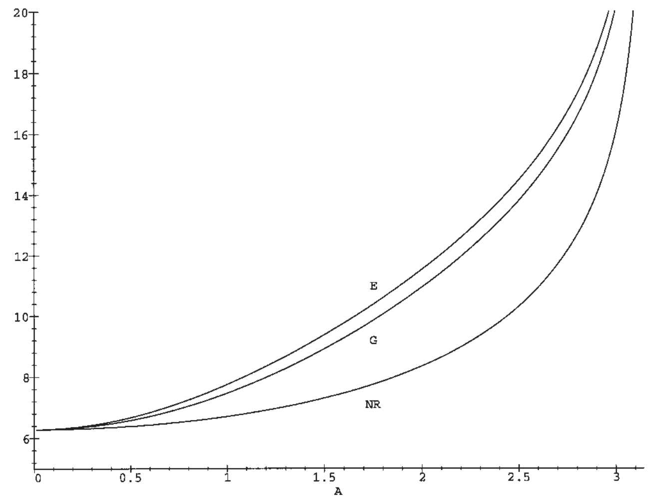

Figure 3
Period of the pendulum vs amplitude: non-relativistic, (NR); relativistic gravity driven, (G); relativistic electric field driven, (E). The vertical axis is the dimensionless period that is determined with respect to a stationary inertial frame.

explicit. Thus, we make a change of the dependent variable:

$$\Omega(\theta) = \frac{d\theta}{d\eta} \ ,$$
$$\frac{d^2\theta}{d\eta^2} = \frac{d\Omega}{d\eta} = \frac{d\Omega}{d\theta}\frac{d\theta}{d\eta} = \Omega\,\frac{d\Omega}{d\theta} \ , \qquad (20)$$

In this new dependent variable, the Equations 16 become:

$$\Omega\,\frac{d\Omega}{d\theta} = -\kappa \sin(\theta)\,[1-\Omega^2] \ , \quad (G)$$
$$\Omega\,\frac{d\Omega}{d\theta} = -\kappa \sin(\theta)\,[1-\Omega^2]^{\frac{3}{2}} \ , \quad (E) \qquad (21)$$

Equations 21 are separable and thus may be integrated. Recalling the initial conditions θ(0) = *A* and Ω(θ(0)) = 0, we obtain for the first integral:

$$\frac{d\theta}{d\eta} = \Omega = -\sqrt{1-\exp\left[2\kappa\{\cos(A)-\cos(\theta)\}\right]} \ , (G)$$
$$\frac{d\theta}{d\eta} = \Omega = -\sqrt{1-\left[1+\kappa\{\cos(\theta)-\cos(A)\}\right]^{-2}} \ . (E) \qquad (22)$$

The period, *P*, is obtained by one more integration of Equations 22:

$$P = 4\int_0^A \frac{d\theta}{\sqrt{1-\exp\left[2\kappa\{\cos(A)-\cos(\theta)\}\right]}} \ , \quad (G)$$

$$P = 4\int_0^A \frac{d\theta}{\sqrt{1-\left[1+\kappa\{\cos(\theta)-\cos(A)\}\right]^{-2}}} \ , \quad (E) \qquad (23)$$

The integrals in Equations 23 were evaluated using Gaussian quadrature. [6] Figure 3 is a plot of the period obtained from Equations 23 for each type of pendulum for a range of amplitudes from 0 to 3 radians and a value of $\kappa = 1$. For comparison, we also plot period for the same amplitude ranges for the non-relativistic pendulum. The period, *P*, does not differ greatly between the two types of relativistic pendula, but the periods of these two pendula do differ from the period of the classical pendulum with the same amplitude. In both relativistic cases, the period is somewhat larger than the classical value. This is the case ecause the effective relativistic mass is greater than its classical couinterpart.

Period in the Reference Frame of the Pendulum Bob

The period values thus far calculated are those determined by observers at rest with respect to the pendulum support, reference frame 'O'. It is possible to calculate the period as determined by observers who are riding in the non-inertial reference frame of the pendulum mass itself. This calculation is accomplished by working in an inertial reference frame which is moving instantaneously with the pendulum mass at some fixed instant of time. An infinitesimal interval of time in this frame is related to the infinitesimal interval of time in the 'O' frame by:

$$d\tau = \frac{dt}{\gamma} = \sqrt{1 - \frac{v^2(t)}{c^2}}\; dt\,, \qquad (24)$$

where *v(t)* is the instantaneous velocity of the pendulum in the 'O' frame. Changing the variable as defined in Equation 20, dimensionless proper time, $d\sigma$, becomes:

$$d\sigma = \sqrt{1 - \Omega^2(\eta)}\; d\eta = \sqrt{1 - \Omega^2(\theta)}\; \frac{d\theta}{\Omega}\,. \qquad (25)$$

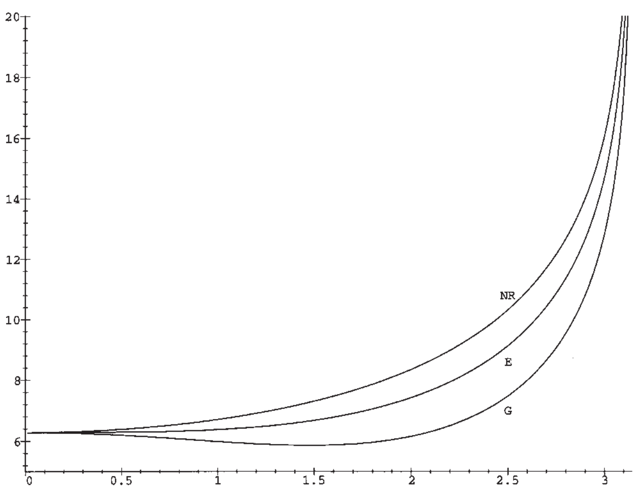

Figure 4
Period of the pendulum vs amplitude: non-relativistic, (NR); relativistic gravity driven, (G); relativistic electric field driven, (E). The vertical axis is the dimensionless period that is determined with respect to the non-inertial frame of the pendulum mass.

The period in the frame of the pendulum can now be found by integrating Equation 25 with respect to the variable θ. Explicit expressions for $W(\theta)$, given in Equations 22, are substituted into Equation 25, resulting in:

$$P = 4\int_0^A \frac{\exp\left[\kappa\,\{\cos(A) - \cos(\theta)\}\right]}{\sqrt{1 - \exp\left[2\,\kappa\,\{\cos(A) - \cos(\theta)\}\right]}}\, d\theta, \quad (G)$$

$$P = 4\int_0^A \frac{d\theta}{\sqrt{\left[1 + \kappa\,\{\cos(\theta) - \cos(A)\}\right]^{-2} - 1}}\,. \quad (E) \qquad (26)$$

The integrals in Equations 25 were again evaluated using Gaussian quadrature and the numerical results plotted in Figure 4. The curves represent the period vs amplitude lie well below the corresponding curves for the non-relativistic pendulum. The two period curves, derived from the different driving forces, are somewhat more separated than the corresponding curves determined for the inertial frame 'O'.

SUMMARY

We have extended the concept of the simple plane pendulum into the relativistic domain. There are two distinct forms of the simple plane pendulum as distinguished by the nature of the force that drives the pendulum. The difference arises because the gravitational driving force is velocity dependent, while the electric driving force is independent of velocity. We solve the equations of motion numerically and calculated the amplitude dependence of the period of each pendulum type in both an inertial frame and in a reference frame moving with the pendulum.

The solution of the relativistic pendulum required no essentially new techniques. The resulting equations are only a bit more difficult to solve than those in the classical case. It surprised us somewhat that we were unable to locate the explicit solutions to the angular motion of the relativistic pendulum. A literature search revealed explicit solutions of the equation of motion for the relativistic simple harmonic oscillator in one dimensions [7,8] which is one component of the two-dimensional motion of the plane pendulum. However, we did not find an explicit solution for the angular motion of such a plane relativistic pendulum. We also found that there remains considerable interest in the pendulum in both relativistic and general relativistic contexts. [9]

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